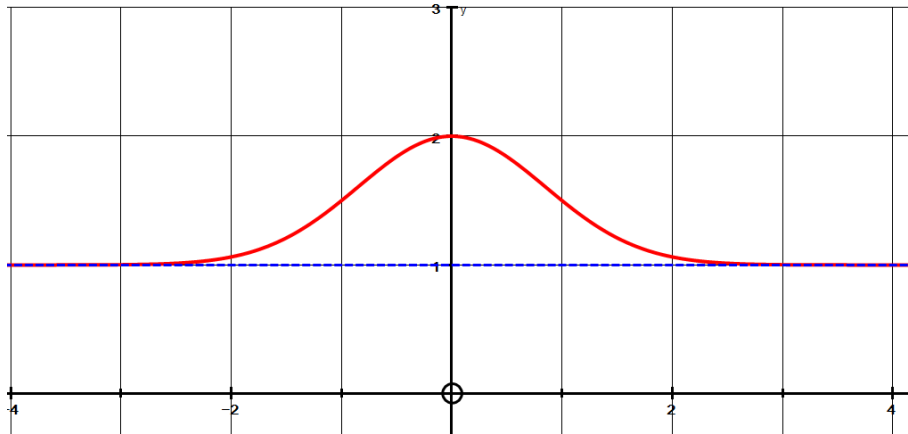


A Brief Summary of ASYMPTOTES.

Def: Asymptote: a line that draws increasingly nearer to a curve without ever meeting it.

There are a basically three types of asymptotes: *horizontal, vertical and oblique.*

1. The graph $y = 1 + 2^{-x^2}$ or $y = 1 + \frac{1}{2^{x^2}}$ has **ONE horizontal asymptote.**



When x is “large” (meaning in this case, $x > 3$ and $x < -3$)

the value of 2^{-x^2} or $\frac{1}{2^{x^2}}$ gets very close to 0

eg if $x = 4$ then 2^{-4^2} or $\frac{1}{2^{4^2}} \approx 0.000015$

so we say that $y \rightarrow 1 + 0 = 1$

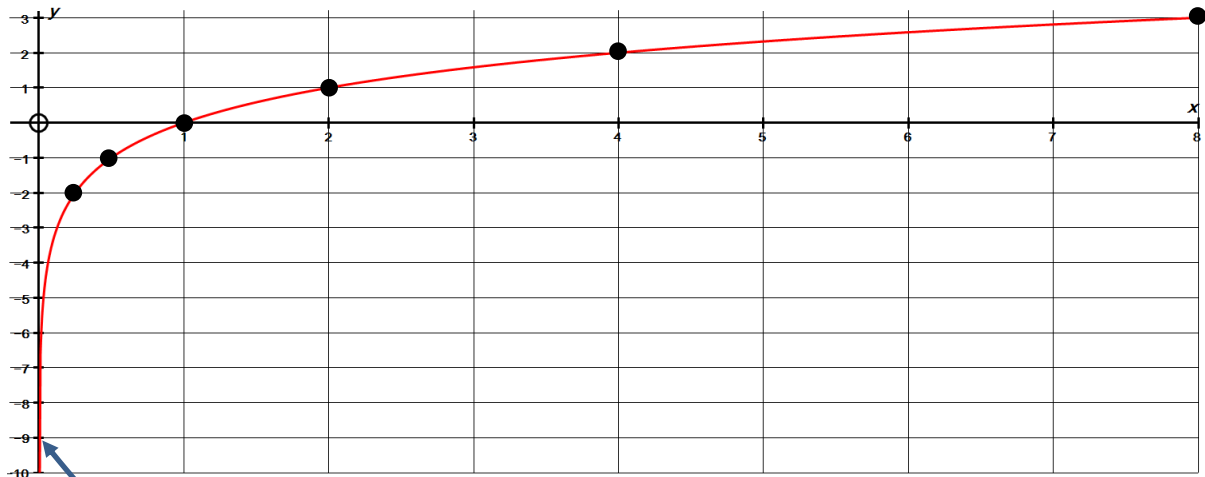
and we say the **horizontal asymptote** is the line $y = 1$

2. A graph can also have just **ONE vertical asymptote.**

Consider $y = \log_2 x$

Typical points are:

x	1	2	4	8	$\frac{1}{2} = 2^{-1}$	$\frac{1}{4} = 2^{-2}$	$\frac{1}{8} = 2^{-3}$	2^{-4}	2^{-10}
$y = \log_2 x$	0	1	2	3	-1	-2	-3	-4	-10



Clearly, the vertical asymptote is the y axis (or the line $x = 0$)

3. Consider the graph of $y = \frac{4}{(x-1)}$

If x is large (in this case let x be over, say 40) eg I will choose $x = 41$

then $y = \frac{4}{40} = \frac{1}{10}$

and going in the negative x direction I will choose $x = -39$

then $y = \frac{4}{-40} = -\frac{1}{10}$

We can say that for large positive x values and “large” negative x values $y \rightarrow 0$ and we say the *horizontal asymptote* is the line $y = 0$

However, when x is close to 1 on either side, the y value gets big!

Approaching $x = 1$ from the right:

If $x = 1.1$ then $y = 40$

If $x = 1.001$ then $y = 4000$

If $x = 1.000001$ then $y = 4\,000\,000$

Approaching 1 from the left:

If $x = 0.9$ then $y = -40$

If $x = 0.999$ then $y = -4000$

If $x = 0.999999$ then $y = -4\,000\,000$

Clearly there is no limit to what the y value can reach!

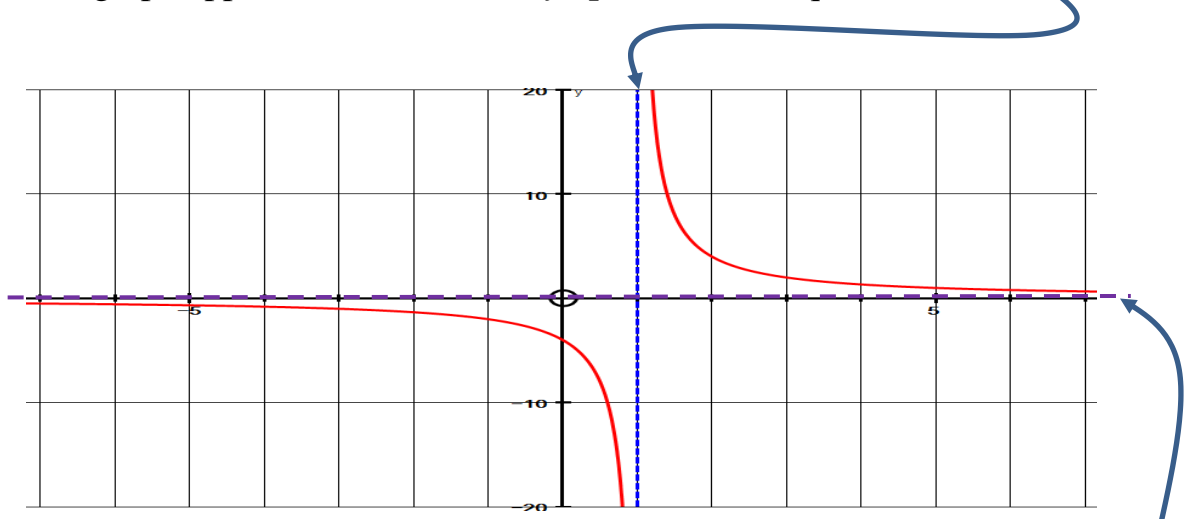
There actually is **no definite number** that y equals when $x = 1$

We say that y is **INFINITE** (*which could be referred to as non finite*).

As $x \rightarrow 1$ from the right we say $y \rightarrow +\infty$

As $x \rightarrow 1$ from the left we say $y \rightarrow -\infty$

The graph approaches a *vertical asymptote* and its equation is $x = 1$



horizontal asymptote $y = 0$

4. It is quite common to see graphs with **vertical** and **horizontal** asymptotes:

eg $y = \frac{2x - 4}{x - 1}$

The vertical asymptote is when the denominator = 0 ie if $x = 1$

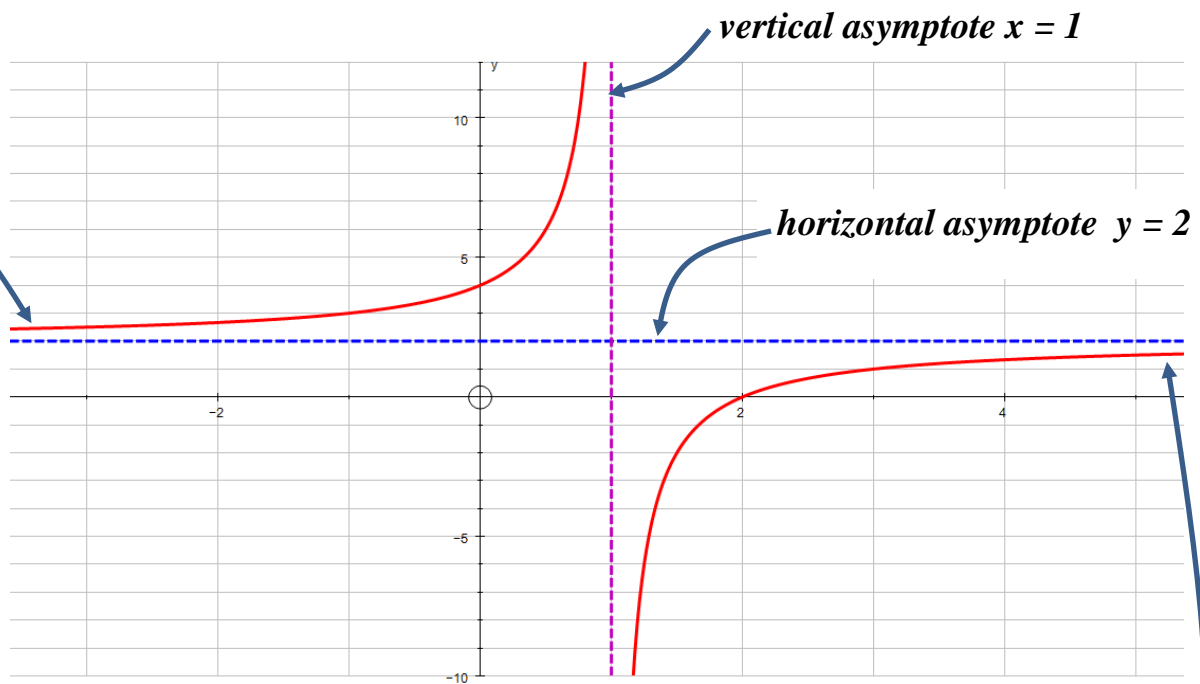
To find the horizontal asymptote we first do a long division:

$$\begin{array}{r} 2 \\ x-1 \overline{) 2x-4} \\ \underline{2x-2} \\ -2 \end{array}$$

so $y = 2 - \frac{2}{x-1}$

When x is large, the term: $\frac{2}{x-1}$ approaches 0

so the horizontal asymptote is $y = 2$



NB A useful idea to remember is if $y = 2 - \frac{2}{x-1} = 2 - \text{"a bit"}$

then for large and positive x values the y value is "a bit less" than 2
so the curve is UNDER the horizontal asymptote $y = 2$

(But for "large" x values which are negative $y = 2 + \text{"a bit"}$ so the left hand side of the graph is ON TOP of the asymptote $y = 2$)

5. The following graph has a **vertical** asymptote and an **oblique** one:

$$y = \frac{x^2}{(x-1)}$$

Firstly, the vertical asymptote is when the denominator is zero so the equation is $x = 1$

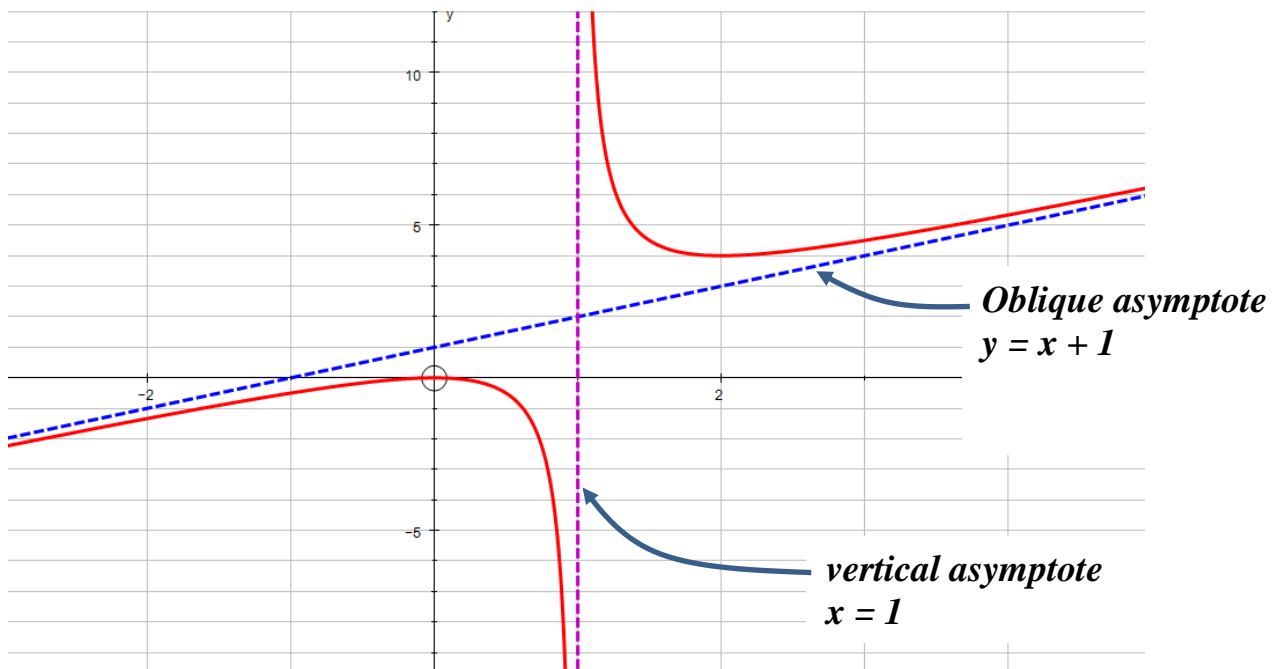
To find the OBLIQUE asymptote we first do a long division:

$$\begin{array}{r} x-1 \overline{) x^2 + 0x + 0} \\ \underline{x^2 - x} \\ x \\ \underline{x-1} \\ 1 \end{array}$$

$$\text{so } y = x + 1 + \frac{1}{x-1}$$

Since $\frac{1}{x-1} \rightarrow 0$ when x is large

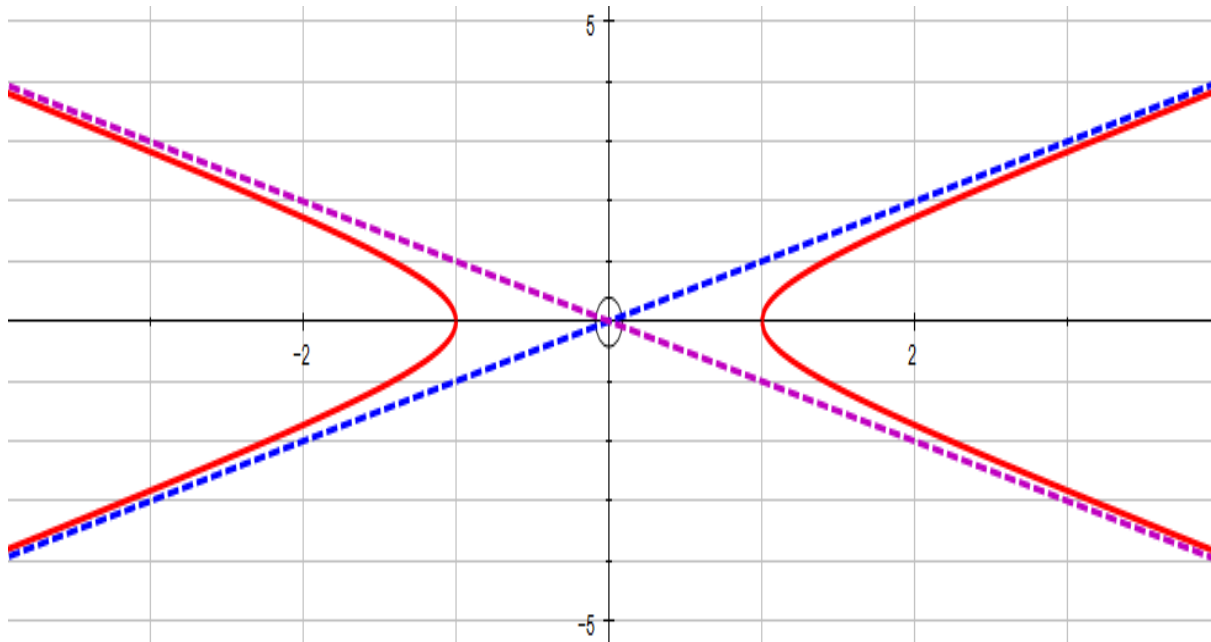
the curve is approximately $y = x + 1$
 This means the curve approaches this line which is referred to as **oblique** because it is neither vertical nor horizontal.



6. A curve can have **TWO** *oblique asymptotes*:

Consider the curve: $y^2 = x^2 - 1$

If x is large then the “1” becomes negligible so the equation reduces to:
 $y^2 \approx x^2$ so that *the asymptotes are $y = \pm x$ as shown below*:



7. I had never come across a curve with a **horizontal** asymptote and an **oblique** one. So I made up one!

Consider $(y - x - 2)^2 = x^2 - 1$

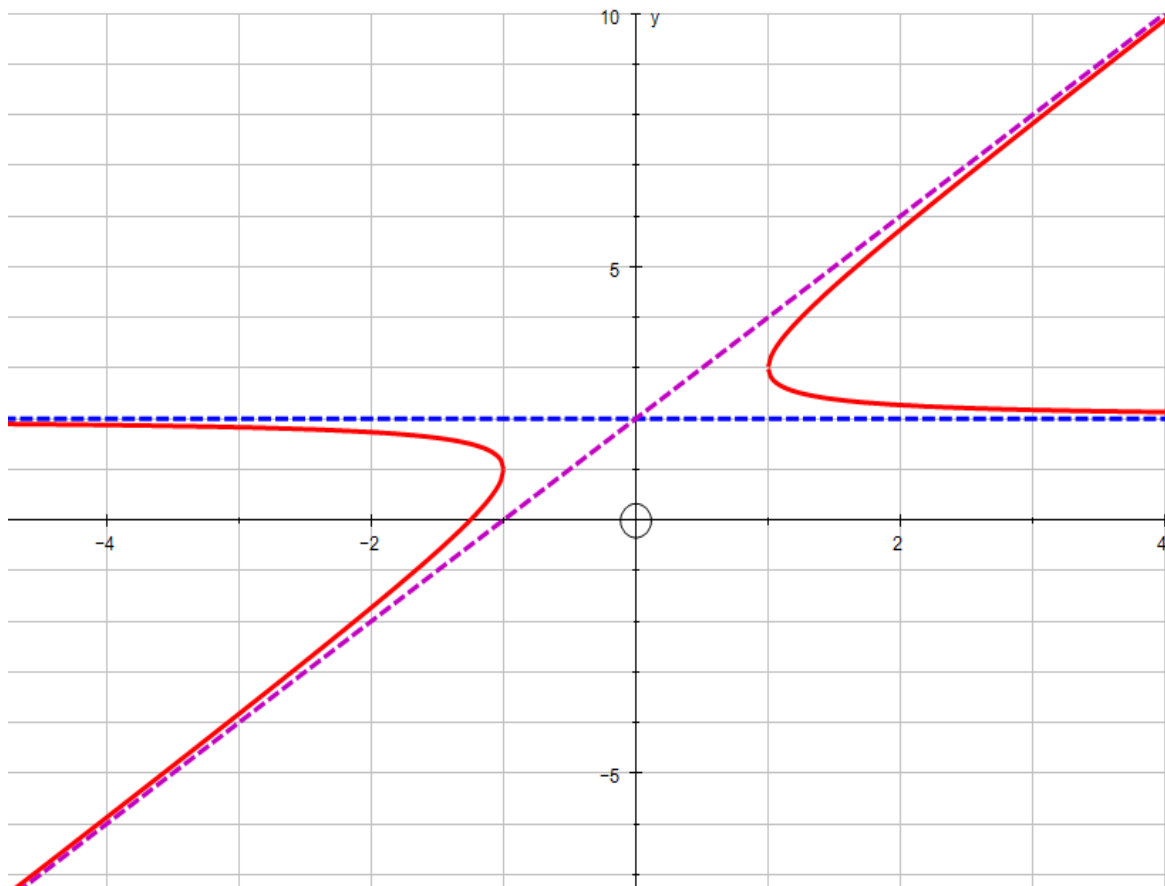
As in number 6, if x and y are large then the “1” becomes negligible so the equation reduces to: $(y - x - 2)^2 \approx x^2$

So $y - x - 2 = +x$ or $y - x - 2 = -x$

$y = 2x + 2$ or $y = 2$

The equation: $y = 2x + 2$ is the oblique asymptote and

The equation: $y = 2$ is the horizontal asymptote.

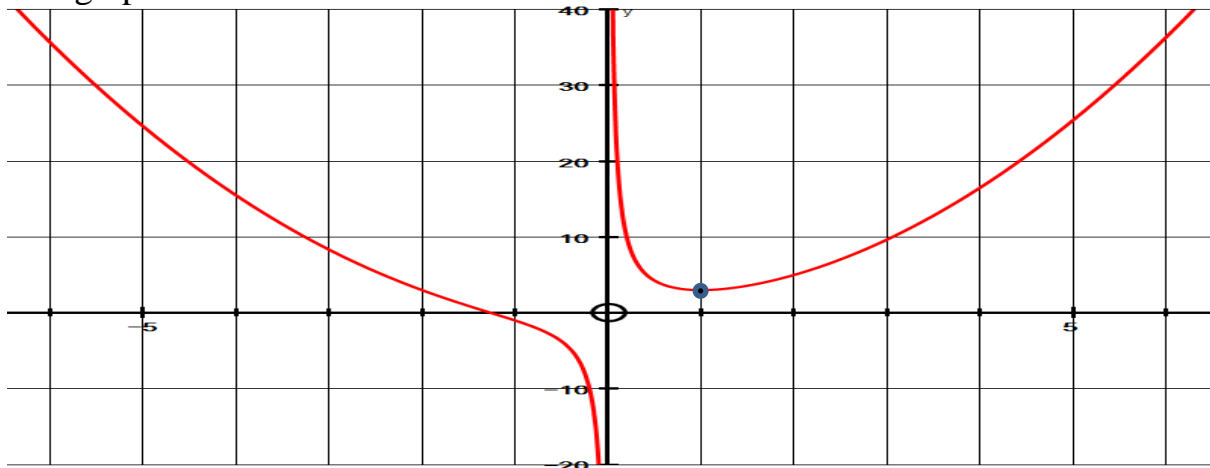


8. Although an asymptote is really a LINE which a curve approaches, we can also have curves approaching **curves!**

For instance, consider $y = \frac{x^3 + 2}{x} = x^2 + \frac{2}{x}$

The denominator is zero if $x = 0$ so this is the *vertical asymptote*.

The graph looks like this:



Just as a matter of interest $y' = 2x - \frac{2}{x^2} = 0$ at the *minimum point*.

$$2x = \frac{2}{x^2}$$

$$\text{so } x^3 = 1$$

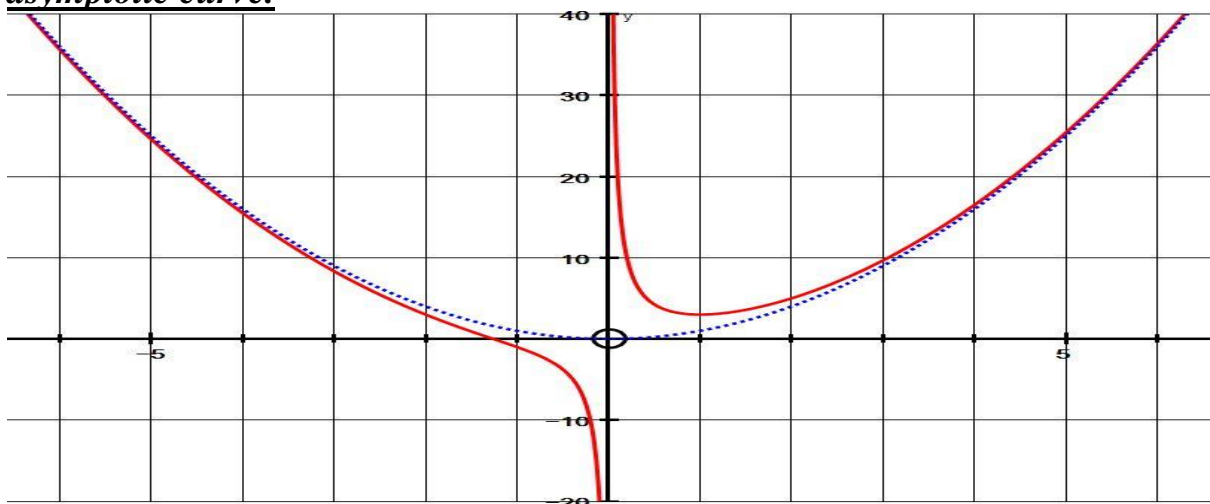
$$x = 1 \text{ and } y = 3 \text{ min point is } (1, 3)$$

However, if we examine: $y = x^2 + \frac{2}{x}$

we see that, if x is large, $\frac{2}{x}$ becomes negligible (if $x = 200$ then $y = 200^2 + \frac{1}{100}$)

so that the equation is approximately $y = x^2$

If I superimpose this parabola on the above graph we can see that it acts as an asymptotic curve.



9. Some people misunderstand the dictionary definition of an asymptote.
 (A line that draws increasingly nearer to a curve **without ever meeting it.**)
Sometimes a curve can pass through an asymptote!
 Remember, an asymptote is what a curve approaches for “large” values of x .

Consider $y = \frac{(x-2)(x-4)}{(x-1)(x-3)} = \frac{x^2 - 6x + 8}{x^2 - 4x + 3}$

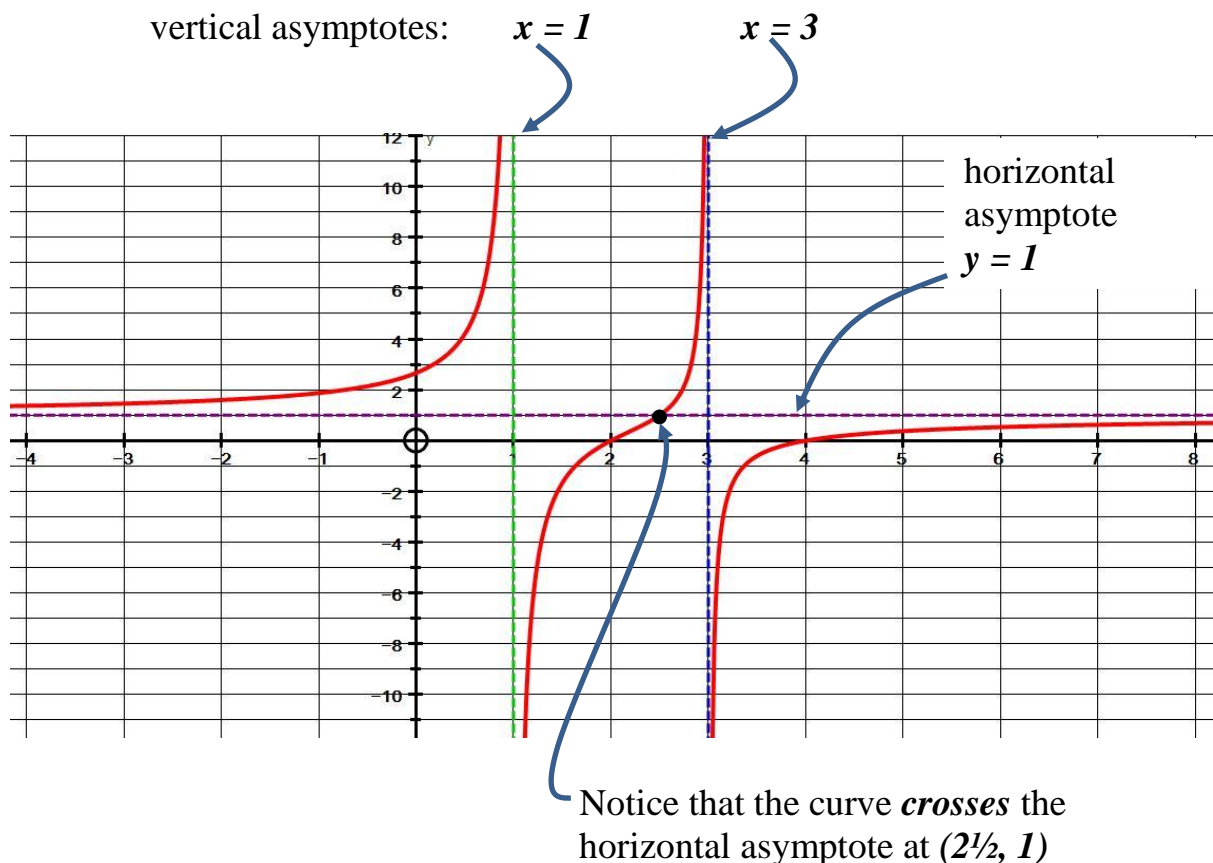
The denominator is zero when $x = 1$ and 3 so the vertical asymptotes are $x = 1$ and $x = 3$

By long division:

$$\begin{array}{r} 1 \\ x^2 - 4x + 3 \overline{) x^2 - 6x + 8} \\ \underline{x^2 - 4x + 3} \\ -2x + 5 \end{array}$$

So $y = 1 + \frac{-2x + 5}{x^2 - 4x + 3}$

and when x is large $y \approx 1$ since $\frac{-2x + 5}{x^2 - 4x + 3} \rightarrow 0$



Note: If $y = \frac{x^2 - 6x + 8}{x^2 - 4x + 3} = 1$ then $x^2 - 6x + 8 = x^2 - 4x + 3$

so $5 = 2x$
 and $x = 2\frac{1}{2}$, $y = 1$

10. We need to be very careful with equations like number 9 because there may be a turning point involved as with the following:

$$y = \frac{x(x-4)}{(x-1)(x-3)} = \frac{x^2 - 4x}{x^2 - 4x + 3} = 1 - \frac{3}{x^2 - 4x + 3}$$

This has the same vertical asymptotes $x = 1$ and $x = 3$ and the same horizontal asymptote $y = 1$ but the graph is quite different!

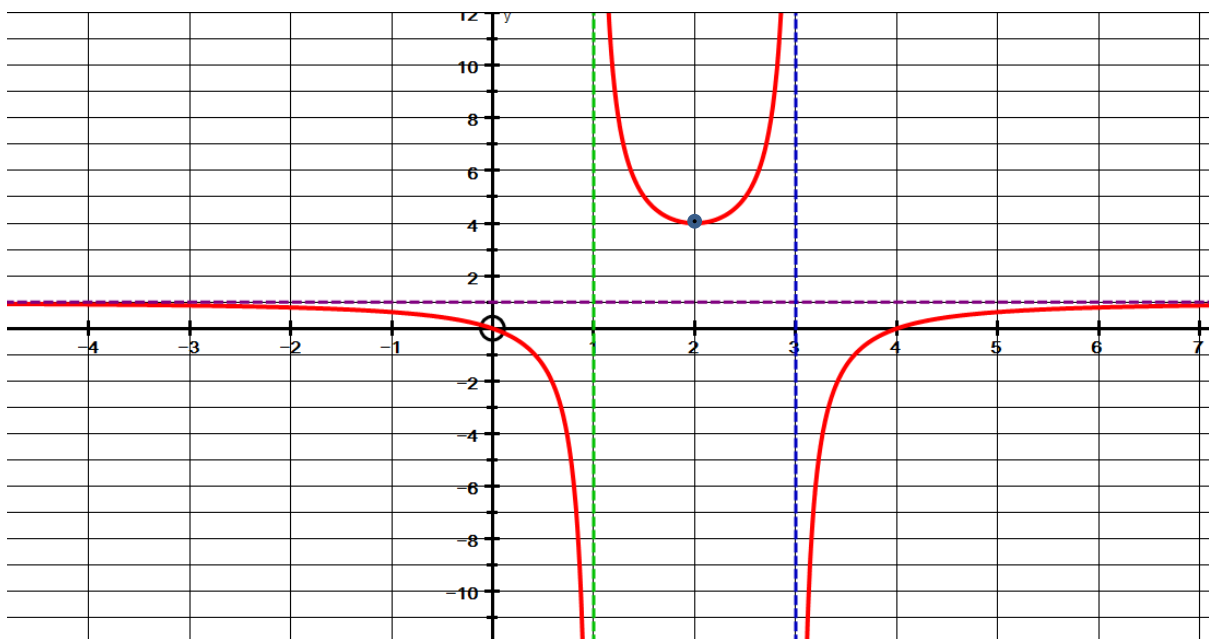
$$y' = \frac{(x^2 - 4x + 3)(2x - 4) - (x^2 - 4x)(2x - 4)}{(x^2 - 4x + 3)^2} = 0 \text{ at max/min points}$$

$$2x^3 - 4x^2 - 8x^2 + 16x + 6x - 12 - 2x^3 + 12x^2 - 16x = 0$$

$$6x - 12 = 0$$

$$x = 2 \text{ and } y = 4$$

minimum point is at (2, 4)



NB A useful idea to remember is: if $y = 1 - \frac{3}{x^2 - 4x + 3} = 1 - \text{“a bit”}$

then for large and positive x values the y value is “a bit less” than 1 so the curve is UNDER the horizontal asymptote $y = 1$
(In this case it is also true for “large” x values which are negative.)

11. There can be a “trap” in the method if we are not fully aware of it.

$$\text{Consider } y = \frac{x^2 - x - 6}{x - 3}$$

We may be *expecting a vertical asymptote with equation $x = 3$* because this makes the denominator zero.

It is true that a denominator of zero makes the number infinite but in this case it makes the numerator zero as well, producing $\frac{0}{0}$ which is not infinite but

it is called *indeterminate*. (ie cannot be determined or found)

If we examine $\frac{x^2 - x - 6}{x - 3}$ and factorise the numerator (ie top line)

$$\text{we get } \frac{(x - 3)(x + 2)}{(x - 3)}$$

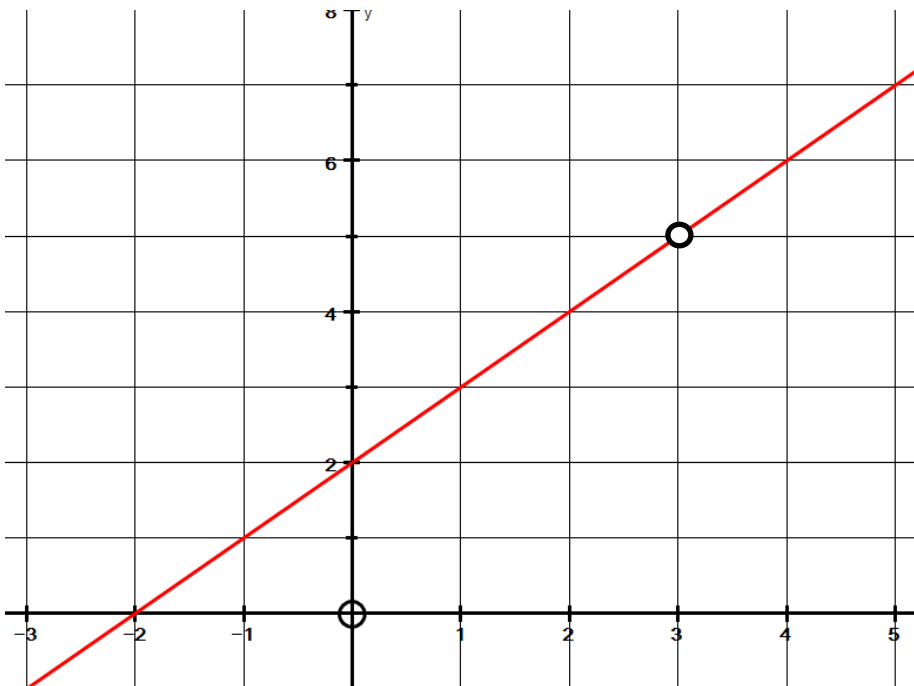
Now this *does* equal $(x + 2)$ if we cancel the $(x - 3)$ terms but we can only do this if $x \neq 3$ because this would involve $\frac{0}{0}$

$$\text{The conclusion is } y = \frac{x^2 - x - 6}{(x - 3)} = \frac{(x - 3)(x + 2)}{(x - 3)} = x + 2 \quad (\text{if } x \neq 3)$$

This means that the graph $y = \frac{x^2 - x - 6}{x - 3}$

is just a line graph with a HOLE in it at $x = 3$.

We show the “hole” by putting an “open circle” at the point $(3, 5)$



12. A very unusual graph is $y = \frac{4x}{x^2 + 4}$

The denominator $x^2 + 4$ cannot be zero so there are **no vertical asymptotes**.

However if x is quite large (in the positive and negative directions)

the expression $\frac{4x}{x^2 + 4}$ **does approach zero**.

eg if $x = 100$ then $y = \frac{400}{160004} \approx 0.0025$

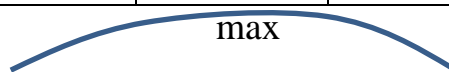
Differentiating: $y = \frac{4x}{x^2 + 4}$

we get $y' = \frac{(x^2 + 4)4 - 4x \cdot 2x}{(x^2 + 4)^2} = 0$ at max/min points

Solving: $4x^2 + 16 - 8x^2 = 0$
 $4x^2 = 16$
 $x^2 = 4$
 $x = \pm 2$

Testing gradients on either side of ± 2

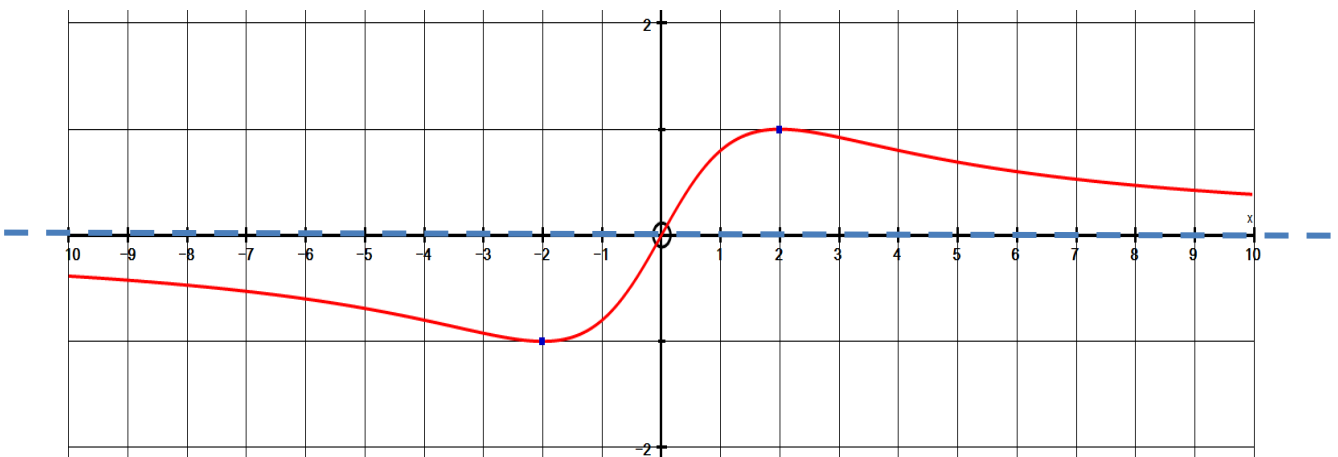
x	1	2	3
y'	+	0	-



x	-3	-2	-1
y'	-	0	+



This is the graph with a max at $(2, 1)$ and a min at $(-2, -1)$ and approaching $y = 0$ in both directions. (note: it crosses the asymptote at $x = 0$)

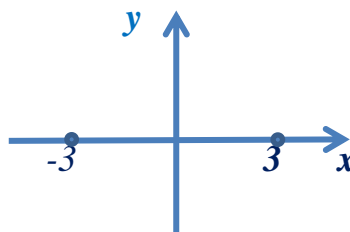


Special Notes for “HYPERBOLAPHOBES”!

13. Consider the Hyperbola: $\frac{(x-6)^2}{9} - \frac{(y+1)^2}{4} = 1$

I usually tell students to first draw $\frac{x^2}{9} - \frac{y^2}{4} = 1$

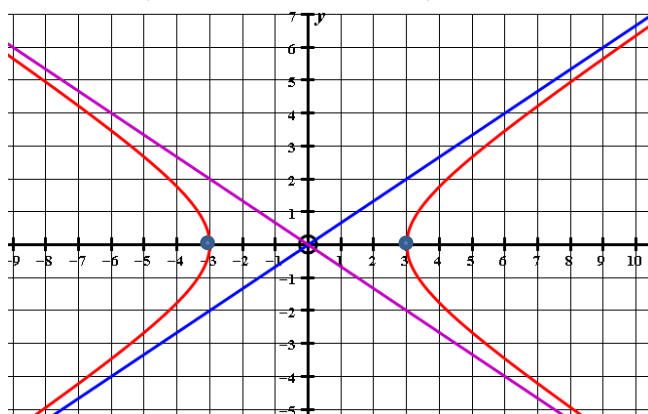
METHOD: Put $y = 0$ so $\frac{x^2}{9} = 1$
 so $x^2 = 9$
 and $x = \pm 3$



Asymptotes : write $\frac{x^2}{9} - \frac{y^2}{4} = 1$
 as $\frac{x^2}{9} = \frac{y^2}{4} + 1$ or $\frac{y^2}{4} + 1 = \frac{x^2}{9}$

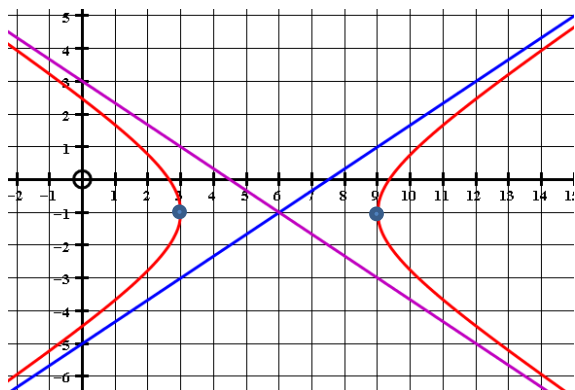
IF x and y are large then the “1” is negligible so:

$\frac{y^2}{4} \approx \frac{x^2}{9}$ so that $y^2 \approx \frac{4}{9}x^2$ and the asymptotes are $y = \pm \frac{2}{3}x$



Note: I never like to use the standard formulae for these asymptotes. The above method emphasises logical thinking.

So to draw $\frac{(x-6)^2}{9} - \frac{(y+1)^2}{4} = 1$ we just move it **along 6** and **down 1**.



We need to move the asymptotes so that they now go through $(6, -1)$
 If $y = \frac{2}{3}x + c$ and $y = -\frac{2}{3}x + d$

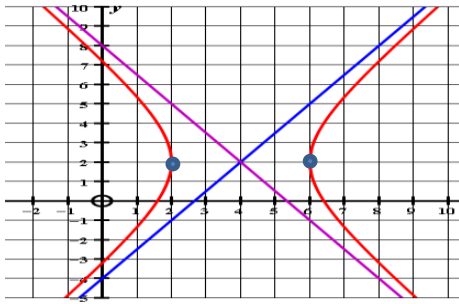
We subs $x = 6, y = -1$

so $c = -5$ and $d = +3$

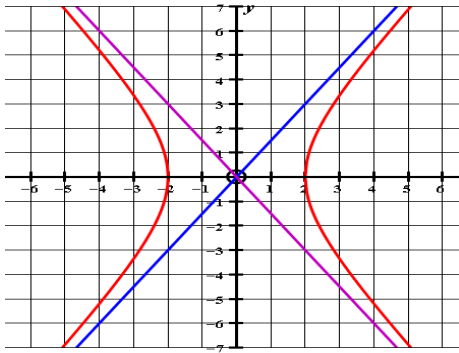
The equations of the asymptotes are:

$y = \frac{2}{3}x - 5$ and $y = -\frac{2}{3}x + 3$

14. AND we should be able to find the equation of an already drawn hyperbola:



Like before it is better to consider the hyperbola with its centre at the origin:



We need to be **sure** the type of equation in this case is :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and not $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ we need to find a and b .

we see the graph goes through $x = 2, y = 0$ so substituting we get :

$$\frac{4}{a^2} = 1 \text{ so that } a^2 = 4$$

The equation so far is $\frac{x^2}{4} - \frac{y^2}{b^2} = 1$

For large x and y (the "1" is negligible) so $\frac{y^2}{b^2} \approx \frac{x^2}{4}$ so $y^2 \approx \frac{b^2}{4}x^2$

and so the asymptotes are $y = \pm \frac{b}{2}x$

BUT WE CAN SEE FROM THE GRAPH THAT THE ASYMPTOTES ARE:

$$y = \pm \frac{3}{2}x$$

SO OBVIOUSLY $b = 3$ and the equation is $\frac{x^2}{4} - \frac{y^2}{9} = 1$

Finally, moving this along 4 and up 2 produces $\frac{(x-4)^2}{4} - \frac{(y-2)^2}{9} = 1$

And if we subs $(4, 2)$ into $y = \frac{3}{2}x + c$ and $y = -\frac{3}{2}x + d$ respectively:

the asymptotes are $y = \frac{3}{2}x - 4$ and $y = -\frac{3}{2}x + 8$