**INTRODUCTION TO PARAMETRIC EQUATIONS**

Suppose the *x* and *y* coordinates of a stone, thrown up in the air, can be calculated at any time *t* seconds using ***x = t and y = 4t – t2***

 ***2***

 ***y***

|  |  |  |
| --- | --- | --- |
| ***t sec*** | ***x*** | ***y*** |
| ***0*** | ***0*** | ***0*** |
|  ***1*** | ***½*** | ***3*** |
| ***2*** | ***1*** | ***4*** |
| ***3*** | ***1 ½*** | ***3*** |
| ***4*** | ***2*** | ***0*** |

 ***0 1 2 x***

The graph has ***x*** and ***y*** axes and ***t*** seems to be an **extra variable** which is not mentioned on the graph.

***t*** is called a PARAMETER.

The equations ***x = t and y = 4t – t2***

 ***2***

are called PARAMETRIC EQUATIONS.

**Finding the gradient.**

 ***x = t and y = 4t – t2***

 ***2***

***dx = ½ and dy = 4 – 2t***

***dt dt***

***dy = dy × dt***

***dx dt dx***

 ***= (4 – 2t) × 2***

 ***1***

 ***= 8 – 4t***

***In some cases we can eliminate the “t” and find the gradient normally:***

***Since t = 2x, we substitute this in y = 4t – t2***

***Producing y = 8x – 4x2***

 ***so dy = 8 – 8x***

 ***dx***

***which is the same as 8 – 4t since t = 2x.***

***However, sometimes we cannot eliminate the parameter so that we must differentiate using the chain rule as above.***

***eg If x = t4 + t and y = t5 – t3***

***dx = 4t3 + 1 and dy = 5t4 – 3t2***

***dt dt***

***dy = dy × dt notice we have dx but we need dt***

***dx dt dx dt dx***

 ***= 5t4 – 3t2***

 ***4t3 + 1***

EXAMPLES:

(1) *If y = et and x = tan(t) find in terms of t*

*(a) dy* ***= et***

 *dt*

*(b) dx* ***= sec2t***

 *dt*

*(c) dy* ***= et***

 *dx* ***sec2t***

(2) A curve is defined parametrically as:

 ***y = 3t2 – 4*** and ***x = t3 + t2***

 Find an expression for ***dy***  at ***t*** = 1

 ***dx***

 ***dy = 6t dx = 3t2 + 2t***

 ***dt dt***

***dy = dy × dt***

***dx dt dx***

 ***= 6t = 6***

 ***3t2 + 2t 5***

(3) The equation of the normal to the

 curve defined parametrically as:

 ***y = e t and x = ln(t)***

 at the point where ***t = 1,*** crosses the ***x***

axis at P. Find the coordinates of P.

***dy = et dx = 1***

***dt dt t***

***dy = dy × dt***

***dx dt dx***

 ***= t et = e if t = 1***

***So grad of normal is –1***

 ***e***

 ***If t = 1, x = 0, y = e***

***Equ of normal is y = mx + c***

 ***e = 0 + c***

***Equ is y = –x + e***

 ***e***

***Crosses x axis when y = 0***

***So x = e2 coords of P = (e2, 0)***

(4)Find the equation of the tangent to the curve : *x = t2 – 4t y = t2 – 2t*

at the point where *t = 3*

***dx = 2t – 4 dy = 2t – 2***

***dt dt***

***dy = 2t – 2 = 4 = 2 if t = 3***

***dx 2t – 4 2***

***x*** **=** ***- 3 and*** ***y = 3 if t = 3***

***tan is y = mx + c***

 ***3 = 2×-3 + c***

 ***c = 9***

***tan is y = 2x + 9***

(5) A curve is defined parametrically by :

 ***y = 6 cos t and x = sin 2t***

Find the gradient of the curve at ***t = π***

 ***6***

 ***dy = –6 sin t dx = –2cos 2t***

 ***dt dt***

 ***dy = dy × dt***

 ***dx dt dx***

 ***= – 6 sin t***

 ***2 cos 2t***

 ***= – 6 × ½ = – 3***

 ***2 × ½***

(6) If ***y = sin(2t)*** and ***x = e4t*** find ***dy***

 ***dx***

 ***when t = 0***

***dy = 2cos 2t dx = 4 e4t***

***dt dt***

***dy = 2cos 2t = 2 = 1***

***dx 4 e4t 4 2***

**(7)** Find the equation of the tangent to the

 curve : ***y = t2 + 6t x = t3 + t***

 at the point where ***t = 1***

***dy = 2t + 6 dx = 3t2 + 1***

***dt dt***

***dy = 2t + 6 = 8 = 2***

***dx 3t2 + 1 4***

***y = 7 , x = 2***

***Equ of tan is of form y = mx + c***

 ***so 7 = 2×2 + c***

 ***3 = c***

***Tangent is y = 2x + 3***

(8) Find *dy* in terms of *t,* for the parametric curve given by *:*

 *dx*

 *y = 3tan(t) , x = 4ln(t)*

 ***dy = 3 sec2t dx = 4***

 ***dt dt t***

 ***dy = dy × dt = 3 sec2t × t***

 ***dx dt dx 4***

(9) For the curve with parametric equations :

 *x = t2 – t y = t2 + t*

 find the equation of the normal to the curve at the point where *t = 2*

***dx = 2t – 1 dy = 2t +1***

***dt dt***

***dy = dy dt = 2t + 1 = 5 grad of normal = – 3***

***dx dt dx 2t – 1 3 5***

***At t = 2 x = 2, y = 6***

***Equ of normal is y = mx + c***

 ***6 = – 3 2 + c c = 36***

 ***5 5***

***Equ of normal is y = – 3 x + 36***

 ***5 5***

(10) A curve is defined parametrically as :

 *y = 4t2 – 3t and x = 3t2 – 4t*

Find expressions for *dy* and *d2y*

 *dx dx2*

***dy = 8t – 3 dx = 6t – 4***

***dt dt***

***dy = dy dt = 8t – 3***

***dx dt dx 6t – 4***

***d2y = dt d 8t – 3 = 1 (6t – 4)8 – (8t – 3)6 = –14***

***dx2 dx dt 6t – 4 6t – 4 (6t – 4)2 (6t – 4)3***

(11) A curve is defined parametrically as :

 *y = t2 – 2t and x = t2 – 4t*

 Find the coordinates and nature of any turning point(s) and show that there

 are no points of inflection.

(i) Find dy = ***2t – 2***

 dt

(ii) Find dx = ***2t – 4***

 dt

(iii) Find dy = ***2t – 2***

 dx ***2t – 4***

(iv) Find t where dy = 0 ***t = 1***

 dx

(v) Use this t value to find the *x* and *y* coordinates of the turning point.

***If t = 1, x = 1 – 2 = – 1 y = 1 – 4 = – 3***

(vi) Calculate ***d2y* using *d dy = dt d dy***

 ***dx2 dx dx dx dt dx***

 ***= 1 (2t – 4)2 – (2t – 2)2***

 ***2t – 4 (2t – 4)2***

 ***= 2t2 – 8 – 2t2 + 4 = – 4***

 ***(2t – 4)3 (2t – 4)3***

(vii) Do 2nd derivative test to see whether the turning point is a max or min.

 ***If t = 1 y′′ = + so MIN point at ( – 1 , – 3)***

(12) A function is defined parametrically as follows:

 *x = t3 + 24t y = t2 – 2t*

(i) Find the *x, y* coordinates of the turning point of this function.

(ii) Find the values of t, *x* and *y* when this function is concave upwards.

***(i) dx = 3t2 + 24 dy = 2t – 2***

 ***dt dt***

***dy = dy × dt = 2t – 2 = 0 at max/min point***

***dx dt dx 3t2 + 24***

 ***2t = 2***

 ***t = 1 so x = 25 and y = –1***

***(b) d2y = dt × d dy = dt × d 2t – 2***

 ***dt2 dx dt dx dx dt 3t2 + 24***

 ***= 1 × (3t2 + 24)2 – (2t – 2)(6t)***

**Must**

**have**

 ***3t2 + 24 (3t2 + 24)2***

 ***= 6t2 + 48 – 12t2 + 12t***

 ***(3t2 + 24)3***

 ***= –6t2 + 12t + 48***

 ***(3t2 + 24)3***

 ***= –6( t2 – 2t – 8)***

 ***(3t2 + 24)3***

 ***= –6(t – 4)(t + 2) = 0 at inflection points so t = 4, –2***

 ***(3t2 + 24)3***

***By 2nd derive test the value of y ′′ at t = 1 is Positive so (25, –1) is a min***

***The function must be concave upwards between points of inflection***

 ***ie for –2 < t < 4 between ( –56, 8) and (160, 8)***

|  |  |  |
| --- | --- | --- |
| ***t*** | ***– 2*** | ***4*** |
| ***x*** | ***– 56*** | ***160*** |
| ***y*** | ***8*** | ***8*** |

 ***Min point (25, –1)***

 ***– 56 160***