**BASICS OF CONCAVITY.**

Concave up

Concave down

5 8

Concave up Concave down Concave up

if x < 5 if 5 < x < 8 if x > 8

6

Concave down if x < 6 Concave up if x > 6

To understand the **2nd Derivative Test**, consider these curves:

**y**

***y = x(x – 3)2***  *MAX*

*INFL*

*= x3 – 6x2 + 9x*

(cubic curve) MIN

1 2 3 4

**y ′**

***y ′ = 3x2 – 12x + 9***

*= 3(x2 – 4x + 3)*

*= 3(x – 1)(x – 3)* 1 2 3 4

(parabola)

**y ′′**

***y ′′ = 6x – 12***

**y ′′ > 0**

(line graph)

1 2 3 4

**y ′′ = 0**

**y ′′< 0**

NOTICE THREE POINTS:

**When the cubic has a MAXIMUM the 2nd derivative is a NEGATIVE number.**

**When the cubic has a MINIMUM the 2nd derivative is a POSITIVE number.**

**When the cubic has an INFLECTION point the 2nd derivative is ZERO.**

**CONCAVITY CHANGES AT INFLECTION POINTS.**

In the following example, the 2nd derivative test fails:

Consider the curve *y = x4 – 4x3 = x3(x – 4)*

*y′ = 4x3 – 12x2*

*= 4x2(x – 3) = 0* for stationary points (max/min/infl)

So *x* = 0 and 3

*y′′ = 12x2 – 24x = 12x(x – 2)*

**2nd derivative test:**

At *x = 3 y′′ = 12x2 – 24x*

*= 36 so MIN at x = 3*

**At *x = 0 y′′ = 12x2 – 24x***

***= 0* ( *neither + nor –* )**

**So we cannot say whether max or min.**

We must now resort to the **1st derivative test** at *x = 0*:

Also, inflection points are where *y′′ = 0*

*12x(x – 2) = 0*

*So x = 0 and 2*

We must now resort to the **1st derivative test** at *x* = 0:

|  |  |  |  |
| --- | --- | --- | --- |
| *x* values | –1 | 0 | 1 |
| gradient | –16 | 0 | –8 |

**Stationary**

**Infl point**

This is a sketch of the curve :

Stationary Infl (0, 0)

1 2 3 4 5

Inflection point (2, –16)

CONCAVE

DOWN Min (3, – 27)

if 0 < x < 2

GENERALLY: The curve *y = axn + bxn – 1  + cxn – 2* +….. has *(n – 1)* turning points.

|  |  |  |
| --- | --- | --- |
| EQUATION | Number of  Turning Points | SHAPE |
| *y = ax2 + bx + c*  *or = (x – p)(x – q)* | 1 |  |
| *y = ax3 + bx2 + cx + d*  *or = (x – p)(x – q)(x – r)* | 2 |  |
| *y = ax4 + bx3 + ……* | 3 |  |

A stationary inflection point counts as **2 turning points combined**.

eg. *y = x3*

Sometimes the above “rule” fails if the gradient cannot equal zero.

eg. *y = x3 – 3x2 + 9x = x(x2 – 3x + 9)* which crosses the *x* axis only at *x = 0*

*y′ = 3x2 – 6x + 9 = 0* at max/min

concave up

so *3(x2 – 2x + 3)* = 0 for x > 1

which has no real solutions 7

so the **gradient cannot equal zero**.

The point of inflection can be found: concave down

for x < 1

*y′′ = 6x – 6 = 0* at infl pt. 0 1 2

so *x = 1 y = 7*

***NB the gradient decreases as we move from x = 0 to x = 1 then it starts to increase again from x = 1 onwards. y′ = 3(x – 1)2 + 6***

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
| ***x*** | 0 | .5 | .9 | **1** | 1.1 | 1.5 | 2 |
| ***y′*** | 9 | 6.75 | 6.03 | **6** | 6.03 | 6.75 | 9 |