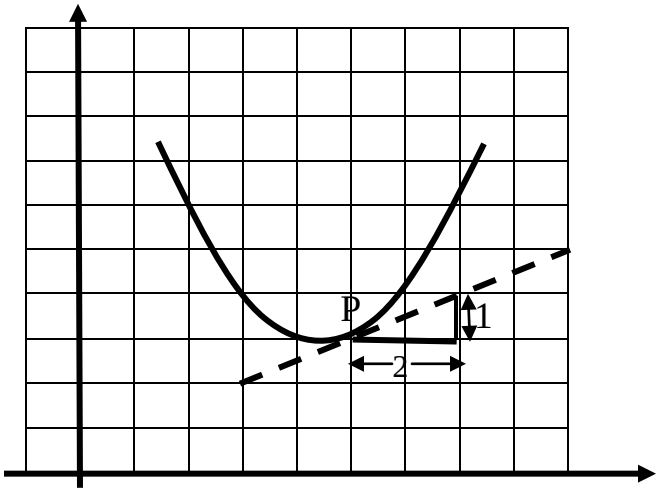
YEAR 13 INTRODUCTION TO CALCULUS.

1. The ideas of Calculus were discovered at the same time by **NEWTON** in England and **LEIBNITZ** in Germany in the 17th century. There was a lot of ill feeling between them because each one wanted to take the credit for discovering Calculus.

2. Newton had a particular interest in the orbits of planets and gravity. He "invented" calculus to help him study such topics.
   His theory was used extensively in putting the first men on the moon and his equations of motion clearly describe the paths of objects thrown through the air.

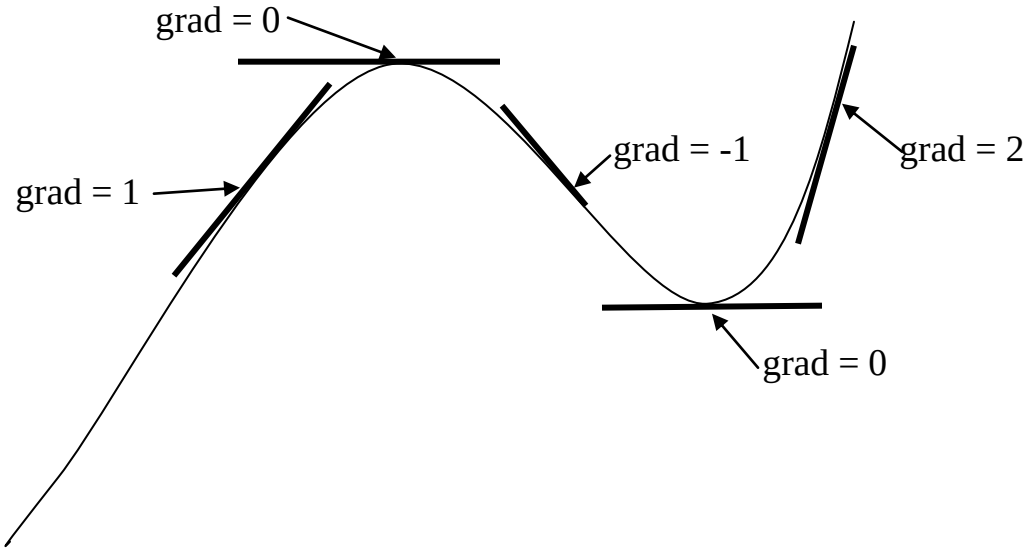
3. The very basic idea of calculus is how to find the changing steepness of curves.

4. Definition: The steepness of a curve at a point P, is the steepness of the TANGENT to the curve at P.

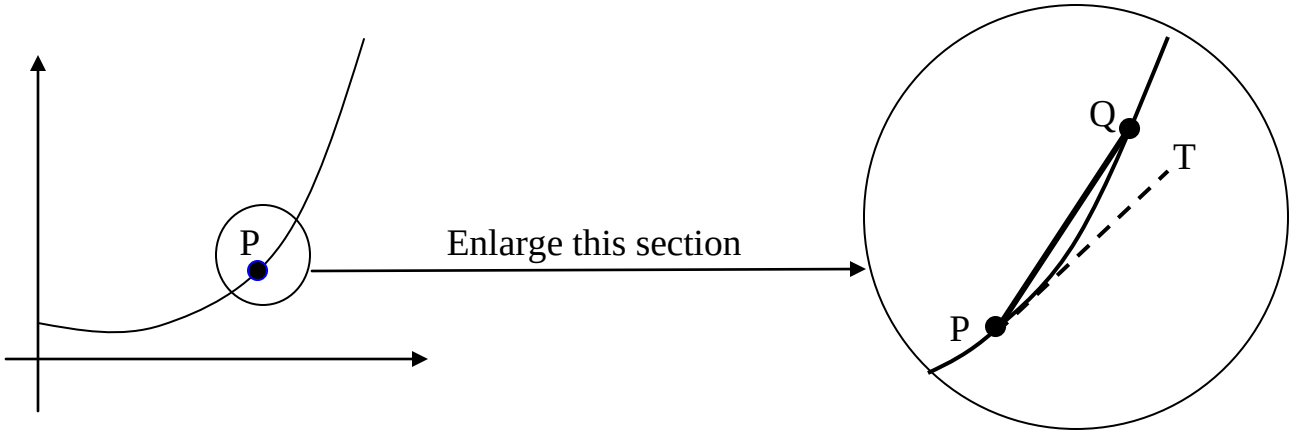


We can see that the gradient of the tangent to the curve, drawn at the point P(5, 3) is clearly $\frac{1}{2}$

5. The **approximate** values of the steepness (or gradient) of the following curve at various points are marked on this diagram:



6. To find the gradient of a curve, we imagine a microscopic section of the curve as shown:



<u>Fundamental Idea</u>: The steepness of tangent PT is approximately equal to the steepness of a chord PQ (where Q is a point on the curve very close to P).

This approximation gets closer to the gradient of the tangent at P by making Q move closer and closer to P.

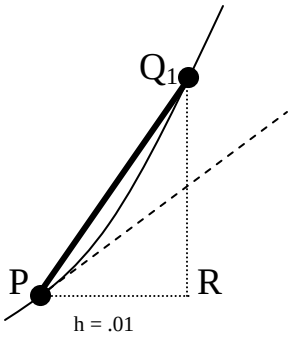


Fig 1.

Grad $PQ_1 = \frac{Q_1R}{PR}$

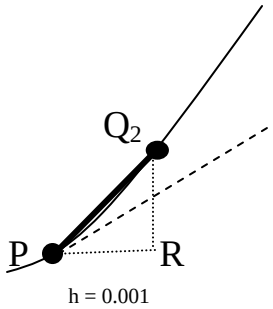


Fig 2.

Grad $PQ_2 = \frac{Q_2R}{PR}$

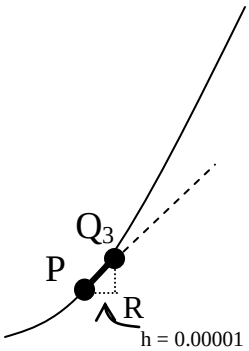


Fig 3.

Grad $PQ_3 = \frac{Q_3R}{PR}$

This geometrical idea has to be changed into mathematical symbols so that we can find the numerical values of these gradients and work out what value they are approaching.

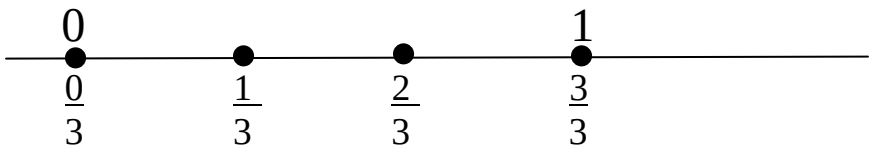
The problem, of course, is that when Q has arrived at P, the value of $\frac{QR}{PR} = \frac{0}{0}$!!!

This relatively simple idea of "finding the gradient of a curve", leads to solving problems such as : finding maximum values of quantities; finding velocities and accelerations; finding the rate at which things increase; finding how high rockets go; finding the positions of planets and stars and many, many more applications.

Brief treatment of $\frac{0}{1}$, $\frac{1}{0}$ and $\frac{0}{0}$

A. Consider the value of $\frac{0}{h}$

Thinking of a simple number line :



Clearly, $\frac{0}{3}$ is "no thirds" and so is zero.

Similarly, $\frac{0}{4} = \frac{0}{9} = \frac{0}{1} = 0$

B. Consider the value of $\frac{1}{0}$

If we think of values of $\frac{1}{h}$ for decreasing values of ***h***, we see what we need.

$$\frac{1}{0.01} = 100$$

$$\frac{1}{0.001} = 1000$$

$$\frac{1}{0.0001} = 10000$$

$$\frac{1}{0.00001} = 100000$$

The smaller ***h*** gets the bigger $\frac{1}{h}$ gets.

We say that as ***h*** approaches zero, $\frac{1}{h}$ becomes "too big to be imagined" or **INfinite**.

The symbol for this "idea" that it is no longer a FINITE number is ∞

C. Consider the value of $\frac{0}{0}$

This is very weird indeed and can be investigated by this odd argument:

If $a \times d = b \times c$ ***then clearly*** $\frac{a}{b} = \frac{c}{d}$

so if $6 \times 0 = 11 \times 0$ ***then similarly*** $\frac{6}{11} = \frac{0}{0}$

Clearly we could have chosen any numbers other than 6 and 11, so we get the weird conclusion that $\frac{0}{0}$ **can equal ANYTHING at all!!**

Actually, we say that $\frac{0}{0}$ is **INDETERMINATE.**

Now this quantity keeps popping up in our theory when trying to find gradients, so we have to learn how to DETERMINE what it does approach.

SIMPLE LIMITS.

Consider the value of $\left(\frac{2^h - 1}{h}\right)$ as h approaches zero.

Clearly we cannot just put $h = 0$ because we get $\frac{2^0 - 1}{0} = \frac{1 - 1}{0} = \frac{0}{0}$!!!

so we choose decreasing values of h and see what the expression approaches as h gets closer to zero.

Let $h = .001$ $\quad \frac{2^{.001} - 1}{.001} = 0.69338745$

Let $h = .0001$ $\quad \frac{2^{.0001} - 1}{.0001} = 0.6931711$

Let $h = .00001$ $\quad \frac{2^{.00001} - 1}{.00001} = 0.693148$

Let h = .000001 $\quad \frac{2^{.000001} - 1}{.000001} = 0.693147$

we can see that this is approaching a value close to ***0.693147***

We write this as $\lim_{h \to 0} \left(\frac{2^h - 1}{h}\right) \approx 0.693147$

Now consider what this limit approaches: $\lim_{\text{as } h \to 0} \dfrac{(3+h)^2 - 9}{h}$

If h = .01 then $\dfrac{(3 + .01)^2 - 9}{.01} = 6.01$

If h = .001 then $\dfrac{(3 + .001)^2 - 9}{.001} = 6.001$

If h = .0001 then $\dfrac{(3 + .0001)^2 - 9}{.0001} = 6.0001$

This is clearly approaching 6

Fortunately, most of the limits we have to deal with can be done using some clever algebra.

Often, we can just examine an expression and SEE what it OBVIOUSLY approaches:

Eg. $\lim_{h \to 0} (4 + h)$ obviously approaches 4 (putting $h = 0$ does not cause a problem)

but for $\lim_{h \to 0} \dfrac{(3 + h)^2 - 9}{h}$ we cannot just put $h = 0$ because we would get : $\dfrac{0}{0}$

We must do some algebraic simplification first :

$$\lim_{h \to 0} \frac{(3 + h)^2 - 9}{h}$$

$$= \lim_{h \to 0} \frac{(9 + 6h + h^2) - 9}{h}$$

$$= \lim_{h \to 0} \frac{6h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(6 + h)}{h}$$

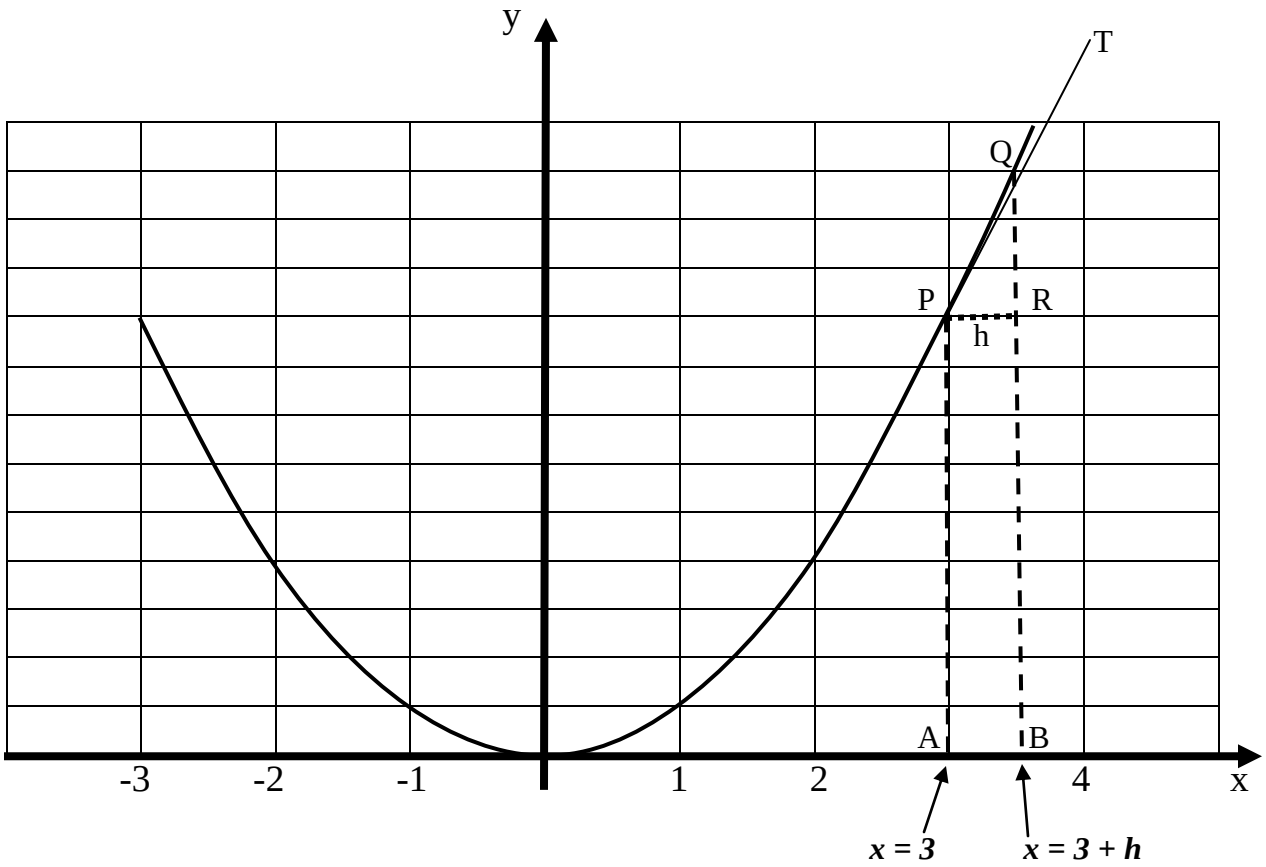
at this stage, h is just a small quantity so that $\dfrac{h}{h}$ *can be cancelled out.*

$$= \lim_{h \to 0} (6 + h)$$

now we just examine this and see what it clearly approaches

$$= 6$$

Now we can apply this to an actual curve such as $y = x^2$ and express this geometrical idea in terms of algebraic symbols to find the gradient at any point, say at $x = 3$.



In this case the gradient of the CHORD PQ already looks very close to the gradient of the TANGENT PT, drawn at P.

Gradient of chord PQ = $\frac{QR}{PR}$ (PR is obviously equal to AB which is h.)

To find QR we must find the length of QB and subtract the length PA.

Using the fact that the equation of the curve is $y = x^2$,

(i) the distance PA is the y value at $x = 3$ so PA = $3^2 = 9$

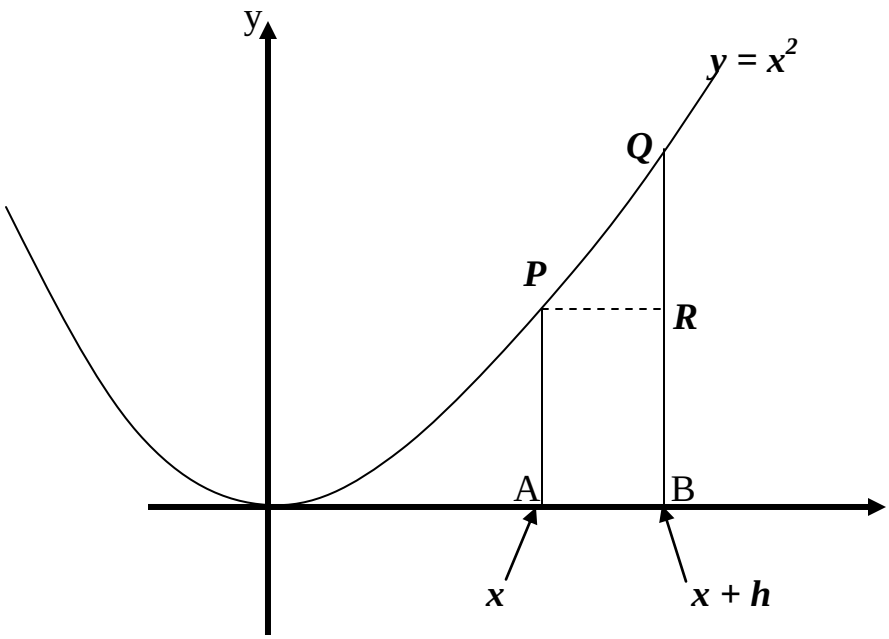
(ii) the distance QB is the y value at $x = 3 + h$ so QB = $(3 + h)^2 = 9 + 6h + h^2$

So the gradient of PQ = $\frac{9 + 6h + h^2 - 9}{h}$

The gradient of the TANGENT is ***"what this approaches"*** (ie the limit as $h \to 0$)

So $\lim_{h \to 0} \frac{6h + h^2}{h}$ = $\lim_{h \to 0} \frac{h(6 + h)}{h}$ = ***6 as we worked out on the previous page.***

It would be silly to do this extensive process every time we need the gradient of the curve so it would make sense to find the gradient at a "general" point ***"x"*** instead of a specific point like ***x = 3 or x = 7*** etc. Then we will have made a "formula" to find the gradient at any particular point we choose!



length of PA = x^2
length of QB = $(x + h)^2$

Gradient of PQ = $\frac{QR}{PR}$
= $\frac{QB - PA}{PR}$
= $\frac{(x + h)^2 - x^2}{h}$

Gradient of tangent at P = $\lim_{h \to 0} \frac{(x + h)^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2}{h}$

$= \lim_{h \to 0} \frac{h(2x + h)}{h}$

$= \lim_{h \to 0} 2x + h$

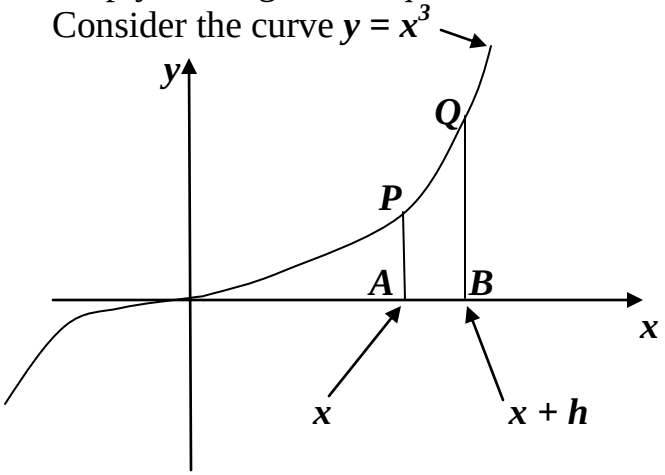
$= 2x$

Using this result, we can find the gradient at ANY ***x*** value.

If x = 3, the gradient is 2×x = 6
If x = 7, the gradient is 2×x = 14
If x = -5, the gradient is 2×x = –10
If x = 0, the gradient is 2×x = 0

We use two symbols for gradient.
Either ***y′*** (Newton's version) or ***$\frac{dy}{dx}$*** (Leibnitz') (version)
Both are useful at different times.

In Year 13 you need to be able to find gradients as above "from 1st principles", as they say.
At this level we just need to find a GENERAL rule for finding the gradient of a curve by simply looking at the equation!
Consider the curve $y = x^3$



As before:
length of PA = x^3
length of QB = $(x + h)^3$

Gradient of tangent at P = $\lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$

$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$

$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$

$= \lim_{h \to 0} 3x^2 + 3xh + h^2$

$= 3x^2$

This process of finding the gradient is called DIFFERENTIATING.

The gradient equation (or the gradient function) is called the DERIVED FUNCTION or the DERIVATIVE.

If we had used $y = x^4$ the gradient would have been $y' = 4x^3$

If we had used $y = x^5$ the gradient would have been $y' = 5x^4$

So the general and incredibly simple rule is:

If $y = x^n$ then gradient is $y' = n x^{n-1}$

Or in words …."**Multiply by the power and reduce the power by one**".
(This rule works for all numerical powers of x.)