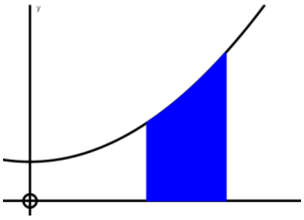
<u>THE FUNDAMENTAL THEOREM OF CALCULUS.</u>

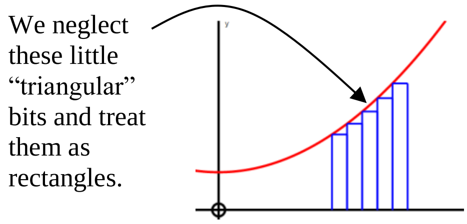
There are TWO different types of **CALCULUS**:

1. **DIFFERENTIATION:** finding **gradients** of curves.
2. **INTEGRATION:** finding **areas** under curves.

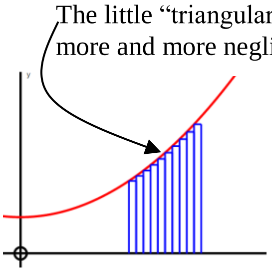
To estimate this area:



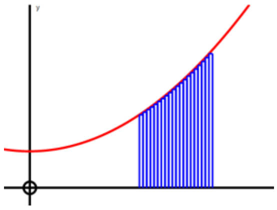
We could split it into strips as below:



To get better approximations we could split the area into <u>more</u> strips:

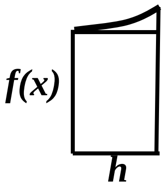


We can see the approximation gets better and better as we use <u>more and more</u> strips:



The sum of the areas of these strips gets closer and closer to the actual area under the curve. We can find this limit as follows:
Consider one strip greatly enlarged for clarity.
We will neglect the curved triangular bit on the top and treat the strip as a rectangle of height $f(x)$ and width h.

Area of 1 strip $\approx f(x) \times h$



Area of all strips $\approx \sum f(x) \times h$

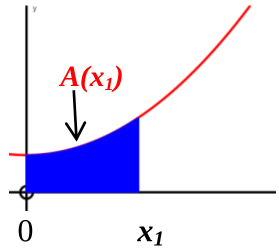
Actual area under the curve $= \lim\limits_{h\to 0} \sum f(x)\times h$ which is written as $\int f(x)\,dx$

> NB At this stage the sign $\int$ only means **the limit of the sum of the strips.**
>
> There is no indication yet that $\int$ has anything to do with **antidifferentiation.**

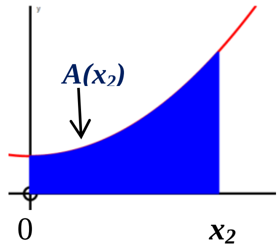
Suppose there exists a "formula" or expression, in terms of *x*, to find the area. (just like there is a formula to find the area of a circle $= \pi r^2$.)

We will call this formula or function $A(x)$.
It is most helpful to think of this in the following way:

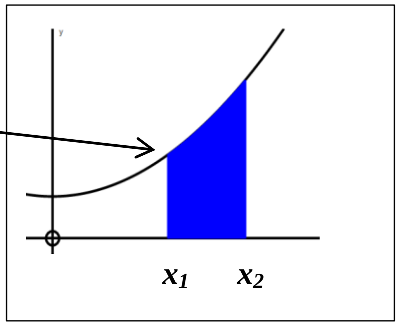
$A(x_1)$ = area under the curve from $x = 0$ *to* $x = x_1$



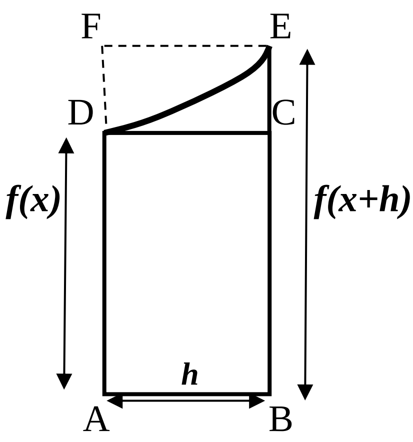
$A(x_2)$ = area under the curve from $x = 0$ *to* $x = x_2$



So the area from x_1 *to* x_2 could be written as $A(x_2) - A(x_1)$



Now consider just one of these strips mentioned earlier. (greatly enlarged)



The area of the strip could be written as :

$$A(x+h) - A(x)$$

The area of the **strip** is between the areas of the rectangles ABCD and ABEF:

area ABCD $< A(x+h) - A(x) <$ **area ABEF**

$$f(x)\times h \quad < \quad A(x+h) - A(x) \quad < \quad f(x+h)\times h$$

Dividing throughout by h, we get:

$$f(x) \quad < \quad \frac{A(x+h) - A(x)}{h} \quad < \quad f(x+h)$$

If we find the **limit** of this as $h \longrightarrow 0$, the three quantities become equal because the two outer quantities both become ***f(x).***

$$\lim_{h\to 0} f(x) \quad < \quad \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \quad < \quad \lim_{h\to 0} f(x+h)$$

So that: $\quad f(x) \quad < \quad \dfrac{dA}{dx} \quad < \quad f(x)$

Obviously $\quad \dfrac{dA}{dx} = f(x)$

In words, this says,

" ***If we* DIFFERENTIATE *the expression for the* AREA *we get the* EQUATION *of the curve*.**"

Or in other words,

"***The formula for* AREA *is the* ANTIDERIVATIVE *of the* equation of the curve."**

Finding the área under a curve using "**the limit of the sum of the strips"** as previously explained, is a very complicated algebraic experience but we have just shown that the sign $\int$ not only means **"the limit of the sum of the strips"** but it also means **antidifferentiate.**

Area A = $\int_a^b f(x)\, dx$ **now means**:

Find the antiderivative of $f(x) = A(x)$ ***then calculate*** $A(b) - A(a)$

eg if $y = f(x) = 3x^2$ and $a = 2$ and $b = 4$ then Area A = $\int_2^4 3x^2 \; dx$

$= \left[x^3 \right]_2^4 \quad = \quad 4^3 - 2^3 = 64 - 8 = 56$ units^2

EDIT. An example of integrating $y = x^2$ from $x = 0$ to $x = 3$ by splitting the area into strips is shown below: