BINOMIAL EXPANSIONS.

I feel that there is no need to use the old traditional "formula method" for finding binomial expansions. I much prefer the following approach.

$(a + b)^0 = 1$
$(a + b)^1 = 1a + 1b$
$(a + b)^2 = 1a^2 + 2a^1b^1 + b^2$
$(a + b)^3 = 1a^3 + 3a^2b^1 + 3ab^2 + b^3$
$(a + b)^4 = 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4$
$(a + b)^5 = 1a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + b^5$

Then, just looking at the numbers (coefficients), we get Pascal's Triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Many years ago, I read that our old friend, Newton, saw a simple pattern for producing these coefficients without having to use Pascal's triangle.

Consider $(a + b)^5$

$$= a^5 + \frac{\mathbf{5}}{\mathbf{1}} a^4b^1 + \frac{\mathbf{5\times4}}{\mathbf{1\times2}} a^3b^2 + \frac{\mathbf{5\times4\times3}}{\mathbf{1\times2\times3}} a^2b^3 + \frac{\mathbf{5\times4\times3\times2}}{\mathbf{1\times2\times3\times4}} a^1b^4 + \frac{\mathbf{5\times4\times3\times2\times1}}{\mathbf{1\times2\times3\times4\times5}} b^5$$

$$= a^5 + \mathbf{5}a^4b + \mathbf{10}\, a^3b^2 + \mathbf{10}\, a^2b^3 + \mathbf{5}\, ab^4 + b^5$$

(Notice the indices of the a's reduce by 1 and the b's increase by 1)

(There is no need to introduce the ${}^nC_r$ notation at all.)

I call this the "thinking method" as opposed to the "formula method".

<u>Special Note</u>: *This "thinking method" applies to indices which are* ***negative*** *or* ***fractional*** *too (but with special restrictions not in this course.)*

$$(1 - r)^{-1} = 1^{-1} + \frac{-1}{1}(-r)^1 + \frac{-1\times-2}{1\times2}(-r)^2 + \frac{-1\times-2\times-3}{1\times2\times3}(-r)^3 + \ldots.$$

$$= 1 + r + r^2 + r^3 + r^4 + \ldots.$$ *which is a geometric series!*

Students can get a better "feel" for this topic using this fundamental method.
Here are some typical problems:

1. Find the coefficient of x^5 in the expansion of $(x + 2)^{10}$

$$(x + 2)^{10} = x^{10} + \frac{10}{1}x^9 2^1 + \frac{10.9}{1.2}.x^8 2^2 + \frac{10.9.8}{1.2.3}.x^7.2^3 + \ldots$$

Clearly by logic!! the term in x^5 will be the 6th term = $\frac{10.9.8.7.6}{1.2.3.4.5}\, x^5.2^5$

So the **coefficient** of x^5 is $\frac{10.9.8.7.6}{1.2.3.4.5}.(2^5) = 8064$

---

2. Find the "CONSTANT" term ie the term independent of x, in the expansion of $(x^2 + \frac{1}{x^2})^6 = (x^2)^6 + \frac{6.}{1}(x^2)^5.(\frac{1}{x^2}) + \frac{6.5}{1.2}(x^2)^4.(\frac{1}{x})^2 + \ldots..$

$$= x^{12} + 6\, x^8 + 15\, x^4 + \ldots\ldots$$

Since the powers are reducing by 4 each time, the constant term is the very next one.

Constant term is the 4th term = $\frac{6.5.4}{1.2.3}.(x^2)^3.(\frac{1}{x^2})^3 = \frac{6.5.4}{1.2.3}.\, x^6\, \frac{1}{x^6} = 20$

---

3. Find the coefficient of x^4 in the expansion of $(2 + \frac{1}{x}).(2x - 3)^6$

$$= (2 + \frac{1}{x})\times\left[(2x)^6 + \frac{6}{1}(2x)^5(-3)^1 + \frac{6.5}{1.2}.(2x)^4(-3)^2 + \frac{6.5.4}{1.2.3}.(2x)^3(-3)^2 + \ldots..\right]$$

Clearly, there are TWO terms with x^4

$$2\times \frac{6.5}{1.2}.(2x)^4(-3)^2 + \frac{1}{x} \times \frac{6}{1}(2x)^5(-3)^1$$

$$= 30\times 16 \times (-3)^2 (x^4) + 6\times 32\times(-3)x^4$$

$$= 4320\, x^4 - 576\, x^4$$

$$= 3744x^4$$

The coefficient is 3744

<u>To show that the sum of the coefficients of a binomial expression is 2^n .</u>

NOTICE THE **SUMS** OF THE COEFFICIENTS OF PASCALL'S TRIANGLE

$(a + b)^0 = 1$
$(a + b)^1 = 1a + 1b$
$(a + b)^2 = 1a^2 + 2a^1b^1 + b^2$
$(a + b)^3 = 1a^3 + 3a^2b^1 + 3ab^2 + b^3$
$(a + b)^4 = 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4$
$(a + b)^5 = 1a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + b^5$

						1							Sum=	1	$=2^0$
					1		1						Sum=	2	$=2^1$
				1		2		1					Sum=	4	$=2^2$
			1		3		3		1				Sum=	8	$=2^3$
		1		4		6		4		1			Sum=	16	$=2^4$
	1		5		10		10		5		1		Sum=	32	$=2^5$
1		6		15		20		15		6		1	Sum=	64	$=2^6$

Consider the general expansion of :

$$(1 + x)^n = 1 + \frac{n}{1} x^1 + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{6} x^3 \ldots + 1$$

LET $x = 1$

$$2^n = 1 + \frac{n}{1} + \frac{n(n-1)}{2} + \frac{n(n-1)(n-2)}{6} \ldots + 1$$

Or $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + {}^nC_4 + \ldots {}^nC_n$