COMBINING DE MOIVRE'S THEOREM AND BINOMIAL EXPANSIONS.

1. Let $z = \text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$
(a) Using De Moivre's theorem find z^3

$$z^3 = \text{cis}(3\theta) = \cos(3\theta) + i\sin(3\theta)$$

(b)Expand $(\cos(\theta) + i\sin(\theta))^3$ using the Binomial theorem

$$z^3 = \cos^3(\theta) + 3\cos^2(\theta)\, i\sin(\theta) + 3\cos(\theta)\,(i\sin(\theta))^2 + (i\sin(\theta))^3$$

$$= \cos^3(\theta) + i3\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

$$= \left(\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)\right) + \left(i(3\cos^2(\theta)\sin(\theta) - \sin^3(\theta))\right)$$

(c) Using parts (a) and (b) find expressions for $\cos(3\theta)$ and $\sin(3\theta)$ in terms of $\sin(\theta)$ and $\cos(\theta)$

Equating real and unreal parts we get :

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) \quad \text{and} \quad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)$$

2. Similarly, find expressions for $\cos(4\theta)$ and $\sin(4\theta)$
Consider $(\cos(\theta) + i\sin(\theta))^4$

(a) using De Moivre's Theorem = $\cos(4\theta) + i\sin(4\theta)$

(b) using Binomial Theorem :

$$= \cos^4(\theta) + 4\cos^3(\theta)\,(i\sin(\theta)) + 6\cos^2(\theta)\,(i\sin(\theta))^2 + 4\cos(\theta)\,(i\sin(\theta))^3 + (i\sin(\theta))^4$$

$$= \left(\cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)\right) + \left(i(4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta))\right)$$

(c) Equating real and unreal parts we get :

$$\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$$

$$\sin(4\theta) = 4\cos^3(\theta)\sin(\theta) - 4\cos(\theta)\sin^3(\theta)$$