ASSIGNMENT N^{0}4. Limits, Basic Differentiation , The Chain Rule and e^x.

1. Evaluate these limits and be sure to use correct mathematical statements.

(a) $\lim_{h\to 0} \dfrac{(5+h)^2 - 25}{h}$

(b) $\lim_{h\to 0} \dfrac{(x+h)^2 - (x-h)^2}{h}$

(c) $\lim_{x\to 4} \dfrac{(x^2 - x - 12)}{(x^2 - 16)}$

(d) $\lim_{h\to 3} \dfrac{(h^2 - 3h)}{(h-3)}$

(e) $\lim_{h\to 0} \dfrac{(h^2 - 3h)}{(h-3)}$

(f) $\lim_{x\to 0} \dfrac{(x^2 - 3x + 2)}{(x^2 - 2x)}$

2. Sometimes we cannot evaluate a limit using algebraic techniques. We have to substitute small values of ***"h"*** such as ***h = 0.001 and h = 0.0001*** and give the answer to a number of significant figures.

Find $\lim_{h\to 0} \dfrac{4^h - 3^h}{h}$ to 3 significant figures.

3. Use the formula :

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

to prove from first principals that the derivative of:

$f(x) = 3x^2 - 5x$

is $f'(x) = 6x - 5$

4. Remembering that the formula stating that if

$y = x^n$ then $\dfrac{dy}{dx} = n\,x^{(n-1)}$

only applies to polynomial functions, find the derivatives of the following.

(a) $y = 6x^3 + 5x^2 - 7x + 1$

(b) $y = 4x^{-6}$

(c) $y = \dfrac{5}{x^2}$

(d) $y = \dfrac{3}{5x^4}$

(e) $y = x^3(4x^2 + 3x - 7)$

(f) $y = x^{2/3}$

(g) $y = \sqrt{x}$

(g) $y = x^{-7/8}$

(h) $y = \dfrac{5}{3\sqrt{x}}$

(i) $y = \dfrac{x^8 + x^5}{x^3} = x^5 + x^2$

(j) $y = \dfrac{x^7 - x^2}{x^4}$

5. If $y = t^5$ and $t = x^3 + 5x^2$

find $\dfrac{dy}{dt}$ and $\dfrac{dt}{dx}$

Now use the "chain rule"

i.e. $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$

to find $\dfrac{dy}{dx}$

6. Use the chain rule to find the derivatives of these functions.

(a) $y = (x^5 + x^4)^7$

(b) $y = (3x - 5)^2$

(c) $y = (4x^2 + 5)^3$

(d) $y = 6(2x - 5)^4$

(e) $y = (x^2 - 3x - 5)^{-2}$

(f) $y = \dfrac{1}{(x-4)}$

(g) $y = \dfrac{5}{(x^2 + 4x - 5)}$

(h) $y = (5x - 4)^{-1/3}$

(i) $y = \sqrt{(4x - 7)}$

(j) $y = e^{4x}$

(k) $y = e^{-2x}$

(l) $y = e^{(5x-2)}$

(m) $y = (e^x + x^2)^4$

(n) $y = \dfrac{1}{4x - e^{3x}}$

(o) $y = \dfrac{1}{e^{2x} - e^{-2x}}$

(p) $y = \dfrac{6}{(x^3 - e^{3x})^2}$

(q)* $y = ((x^2 + 2)^4 + 1)^3$

ANSWERS N^{0}4

1. Evaluate these limits and be sure to use correct mathematical statements.

(a) $\lim_{h\to 0} \frac{(5+h)^2 - 25}{h}$

$$= \lim_{h\to 0} \frac{25+10h+h^2-25}{h}$$
$$= \lim_{h\to 0} \frac{h(10-h)}{h}$$
$$= 10$$

(b) $\lim_{h\to 0} \frac{(x+h)^2 - (x-h)^2}{h}$

$$= \lim_{h\to 0} \frac{x^2+2hx+h^2-x^2+2hx-h^2}{h}$$
$$= \lim_{h\to 0} \frac{4hx}{h}$$
$$= 4x$$

(c) $\lim_{x\to 4} \frac{(x^2 - x - 12)}{(x^2 - 16)}$

$$\lim_{x\to 4} \frac{(x-4)(x+3)}{(x-4)(x+4)} = \frac{(x+3)}{(x+4)}$$
$$= 0.875$$

(d) $\lim_{h\to 3} \frac{(h^2 - 3h)}{(h-3)}$

$$= \lim_{h\to 3} \frac{h(h-3)}{(h-3)} = \lim_{h\to 3} h = 3$$

(e) $\lim_{h\to 0} \frac{(h^2 - 3h)}{(h-3)}$

$$= \lim_{h\to 0} \frac{h(h-3)}{(h-3)} = \lim_{h\to 0} h = 0$$

(f) $\lim_{x\to 0} \frac{(x^2 - 3x + 2)}{(x^2 - 2x)}$

$$= \lim_{x\to 0} \frac{(x-2)(x-1)}{x(x-2)} = \frac{x-1}{x}$$

The limit does not exist.

2.Find $\lim_{h\to 0} \frac{4^h - 3^h}{h}$ to 3 significant figures.

If x=1 expression is 1
x=0.1 expression is 0.326
x=0.01 expression is 0.291
x=0.001 expression is0.288
x=0.0001, exprn is 0.288
The limit is 0.288

3. Use the formula :

$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$

to prove from first principals that the derivative of:

$f(x) = 3x^2 - 5x$

is $f'(x) = 6x - 5$

$$= \lim_{h\to 0} \frac{3(x+h)^2 - 5(x+h) - 3x^2 + 5x}{h}$$
$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 5x - 5h - 3x^2 - 5x}{h}$$
$$= \lim_{h\to 0} \frac{h(6x+3h-5)}{h}$$
$$= 6x - 5$$

4.(a) $y = 6x^3 + 5x^2 - 7x + 1$

$$\frac{dy}{dx} = 18x^2 + 10x - 7$$

(b) $y = 4x^{-6}$

$$\frac{dy}{dx} = -24x^{-7} = \frac{-24}{x^7}$$

(c) $y = \frac{5}{x^2}$

$$\frac{dy}{dx} = \frac{-10}{x^3}$$

(d)

$$y = \frac{3}{5x^4} = \frac{3x^{-4}}{5}$$
$$\frac{dy}{dx} = \frac{-12x^{-5}}{5}$$

(e) $y = x^3(4x^2 + 3x - 7)$

$y = 4x^5 + 3x^4 - 7x^3$

$$\frac{dy}{dx} = 20x^4 + 12x^3 - 21x^2$$

(f) $y = x^{2/3}$

$$\frac{dy}{dx} = -\tfrac{2}{3}x^{-1/3}$$

(g) $y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

(h) $y = x^{-7/8}$

$$\frac{dy}{dx} = \frac{-7}{8}x^{-15/8}$$

(h) $y = \frac{5}{3\sqrt{x}}$

$y = \frac{5}{3}x^{-1/2}$ $\frac{dy}{dx} = \frac{-5}{6}x^{-3/2}$

(i) $y = \frac{x^8 + x^5}{x^3} = x^5 + x^2$

$\frac{dy}{dx} = 5x^4 + 2x$

(j) $y = \frac{x^7 - x^2}{x^4}$

$y = x^3 - x^{-2}$ $\frac{dy}{dx} = 3x^2 + 2x^{-3}$

5. If $y = t^5$ and $t = x^3 + 5x^2$

$\frac{dy}{dt} = 5t^4$

$\frac{dt}{dx} = 3x^2 + 10x$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$\frac{dy}{dx} = 5(x^3 + 5x^2)^4(3x^2 + 10x)$

6. Use the chain rule to find the derivatives of these functions.

(a) $y = (x^5 + x^4)^7$

$\frac{dy}{dx} = 7(x^5 + x^4)^6(5x^4 + 4x^3)$

(b) $y = (3x - 5)^2$

$\frac{dy}{dx} = 2(3x-5)\times 3 = 6(3x-5)$

(c) $y = (4x^2 + 5)^3$

$\frac{dy}{dx} = 3(4x^2 + 5)^2(8x)$

(d) $y = 6(2x - 5)^4$

$\frac{dy}{dx} = 24(2x-5)^3(2) = 48(2x-5)^3$

(e) $y = (x^2 - 3x - 5)^{-2}$

$\frac{dy}{dx} = -2(x^2 - 3x - 5)^{-3}(2x-3)$

(f) $y = \frac{1}{(x - 4)}$

$\frac{dy}{dx} = \frac{-1}{(x-4)^2}$

(g) $y = \frac{5}{(x^2 + 4x - 5)}$

$\frac{dy}{dx} = \frac{-5(2x+4)}{(x^2+4x-5)^2}$

(h) $y = (5x - 4)^{-1/3}$

$\frac{dy}{dx} = -\frac{5}{3}(5x-4)^{-4/3}$

(i) $y = \sqrt{(4x - 7)}$

$\frac{dy}{dx} = \frac{1}{2}(4x-7)^{-1/2} \times 4 = 2(4x-7)^{-1/2}$

(j) $y = e^{4x}$

$\frac{dy}{dx} = 4e^{4x}$

(k) $y = e^{-2x}$

$\frac{dy}{dx} = -2e^{-2x}$

(l) $y = e^{(5x-2)}$

$\frac{dy}{dx} = 5e^{(5x-2)}$

(m) $y = (e^x + x^2)^4$

$\frac{dy}{dx} = 4(e^x + x^2)^3(e^x + 2x)$

(n) $y = \frac{1}{4x - e^{3x}}$

$\frac{dy}{dx} = \frac{-(4-3e^{3x})}{(4x-e^{3x})^2}$

(o) $y = \frac{1}{e^{2x} - e^{-2x}}$

$\frac{dy}{dx} = \frac{-(2e^{2x} + 2e^{2x})}{(e^{2x} - e^{-2x})^2}$

(p) $y = \frac{6}{(x^3 - e^{3x})^2}$

$\frac{dy}{dx} = -12(x^3 - e^{3x})^{-3}(3x^2 - 3e^{3x})$

(q)* $y = ((x^2 + 2)^4 + 1)^3$

$\frac{dy}{dx} = 3((x^2+2)^4 + 1)^2(4(x^2+2)^3 \times 2x)$